Solution of A-D equation with condition 2

Basics of Fourier transform

Fourier transform($\mathcal{F}$) of *f(x)* (-∞<x<∞)
Fourier inverse transform ($\mathcal{F}^{-1}$) of $\mathcal{F}(w)$ (-∞<w<∞)

$$f(x) \Rightarrow \mathcal{F} \Rightarrow F(w) \Rightarrow \mathcal{F}^{-1} \Rightarrow f(x)$$

$$\mathcal{F}[f]=F(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx \quad (1)$$

$$\mathcal{F}^{-1}[F]=f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(w)e^{iwx}dw \quad (2)$$

Fourier transforms of various functions are tabulated.

ex)

$$f(x)=\begin{cases} e^{-x} & (x \geq 0) \\ -e^{x} & (x \prec 0) \end{cases} \Longleftrightarrow F(w)=i\sqrt{\frac{2}{\pi}}\frac{w}{1+w^2} \quad (3\text{-}1)$$

$$f(x)=\begin{cases} 1 & (-1 \prec x \prec 1) \\ 0 & (x \leq -1, x \geq 1) \end{cases} \Longleftrightarrow F(w)=\sqrt{\frac{2}{\pi}}\frac{\sin w}{w} \quad (3\text{-}2)$$

$$f(x)=e^{-x^2} \Longleftrightarrow F(w)=\frac{1}{\sqrt{2}}e^{-(w/2)^2} \quad (3\text{-}3)$$

Basics of Fourier transform cont.

- **linearity of Fourier transform**

  for any functions *f(x)* and *g(x)*, constants *a,b*

$$\mathcal{F}[af+bg]=a\mathcal{F}[f]+b\mathcal{F}[g] \quad (4)$$

- **Fourier transform of the derivative about *x***

OD $$\mathcal{F}[f'(x)]=iw\mathcal{F}[f(x)] \qquad \mathcal{F}[f''(x)]=-w^2\mathcal{F}[f(x)] \quad (5)$$

PD $$\mathcal{F}[C_x]=iw\mathcal{F}[C] \qquad \mathcal{F}[C_{xx}]=-w^2\mathcal{F}[C] \quad (6)$$

$$\mathcal{F}[C_t]=\frac{\partial}{\partial t}\mathcal{F}[C] \qquad \mathcal{F}[C_{tt}]=\frac{\partial^2}{\partial t^2}\mathcal{F}[C] \quad (7)$$

$$C_x=\frac{\partial C(x,t)}{\partial x} \quad C_{xx}=\frac{\partial^2 C(x,t)}{\partial x^2} \quad C_t=\frac{\partial C(x,t)}{\partial t} \quad C_{tt}=\frac{\partial^2 C(x,t)}{\partial t^2}$$

Basics of Fourier transform cont.

•**convolution**

convolution of functions *f(x)* and *g(x)*:

$$(f*g)(x)=\int_{-\infty}^{\infty} f(w)g(x-w)dw=\int_{-\infty}^{\infty} f(x-w)g(w)dw \quad (8)$$

$$\mathcal{F}(f*g)=\sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(g) \quad (9)$$

inverse Fourier transform of (9)

$$(f*g)=\sqrt{2\pi}\mathcal{F}^{-1}\left[\mathcal{F}(f)\mathcal{F}(g)\right] \quad (10)$$

(2) ⟶ $$(f*g)(x)=\int_{-\infty}^{\infty}\mathcal{F}(f)\mathcal{F}(g)e^{iwx}dw \quad (11)$$

Solution of Diffusive equation with condition 2

Heaviside fun.

No B.C.

$$\frac{\partial C}{\partial t}=D_m\frac{\partial^2 C}{\partial x^2} \quad (12)$$

I.C. $C(x,0)=\{1-H(x)\}C_0=\begin{cases}0 & x>=0\\ C_0 & x<0\end{cases}$ (13)

<u>Step1</u>: as *x* can be expanded from $-\infty$ to ∞, make Fourier transforms of (12), (13) about *x*.

$$\mathcal{F}[C_t]=D_m\mathcal{F}[C_{xx}] \quad (14) \qquad \mathcal{F}[C(x,0)]=\mathcal{F}[1-H(x)] \quad (15)$$

↓ (6)(7)

$$\frac{\partial U(t)}{\partial t}=-D_m w^2 U(t) \quad (14') \qquad U(0)=\mathcal{F}[1-H(x)]=\Phi(w) \quad (15')$$

<u>Step2</u>: solve transformed equations (14’)(15’)

U(t)=*$\mathcal{F}$*[C(x,t)] is function of *t* and *w*, but *w* can be treated as const. for DE (14’).

$$U(t)=\Phi(w)e^{-D_m w^2 t} \quad (16)$$

Solution of D equation with condition 2 cont.

Step3: Inverse Fourier transforms of U(t,w) $\Rightarrow$ $C(x,t)$

$$C(x,t)=\mathcal{F}^{-1}\left[U(w,t)\right]=\mathcal{F}^{-1}\left[\Phi(w)e^{-D_m w^2 t}\right] \qquad (17)$$

From convolution theorem (10)

$$C(x,t)=\mathcal{F}^{-1}\left[\Phi(w)e^{-D_m w^2 t}\right]=\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left[\Phi(w)\right]*\mathcal{F}^{-1}\left[e^{-D_m w^2 t}\right]$$

$$=\frac{1}{\sqrt{2\pi}}(1-H(x))C_0*\left[\frac{1}{\sqrt{2D_m t}}e^{-x^2/(4D_m t)}\right]\leftarrow (2),(3\text{-}3)\leftarrow \left(\frac{s}{2}\right)^2=D_m t w^2$$

$$(8)\longrightarrow \quad =\frac{C_0}{2\sqrt{\pi D_m t}}\int_{-\infty}^{\infty}(1-H(w))e^{-(x-w)^2/(4D_m t)}dw \qquad (18)$$

Solution of D equation with condition 2 cont.

$$(13)\longrightarrow(18) \qquad C(x,t)=\frac{C_0}{2\sqrt{\pi D_m t}}\int_{-\infty}^{0}e^{-(x-w)^2/(4D_m t)}dw \qquad (19)$$

$$\beta=\frac{x-w}{2\sqrt{D_m t}},\quad d\beta=\frac{-1}{2\sqrt{D_m t}}dw \qquad (20)$$

$$C(x,t)=\frac{C_0}{2}\int_{x/(2\sqrt{D_m t})}^{\infty}e^{-\beta^2}d\beta \qquad (21)$$

Advective part can be included by coordinate transformation:

$$\xi=x-v_{\text{int}}t$$

$$\tau=t$$

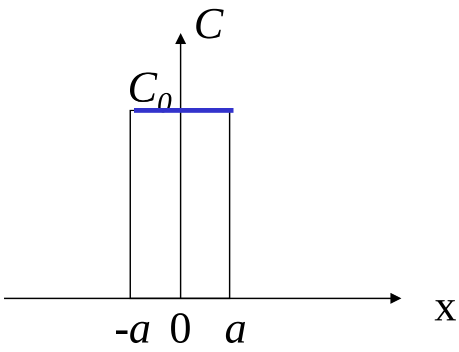
$$=\begin{cases}\dfrac{1}{2}C_0\left[1+erf\,\dfrac{-x}{2(D_m t)^{0.5}}\right] & (x<0)\\[2ex] \dfrac{1}{2}C_0 erfc\,\dfrac{x}{2(D_m t)^{0.5}} & (0\le x)\end{cases} \qquad (22)$$

Solution of different initial condition

No B.C.

I.C. $C(x,0)= C_0 \quad -a < x < a$

$\quad\quad =0 \quad -a > x, \ x > a$ (23)



23) ⟶ (18) $$C(x,t) = \frac{C_0}{2\sqrt{\pi D_m t}} \int_{-a}^{a} e^{-(x-w)^2/(4D_m t)} dw \quad (24)$$

for pulse source ***a*** =>0, *dw=2a*

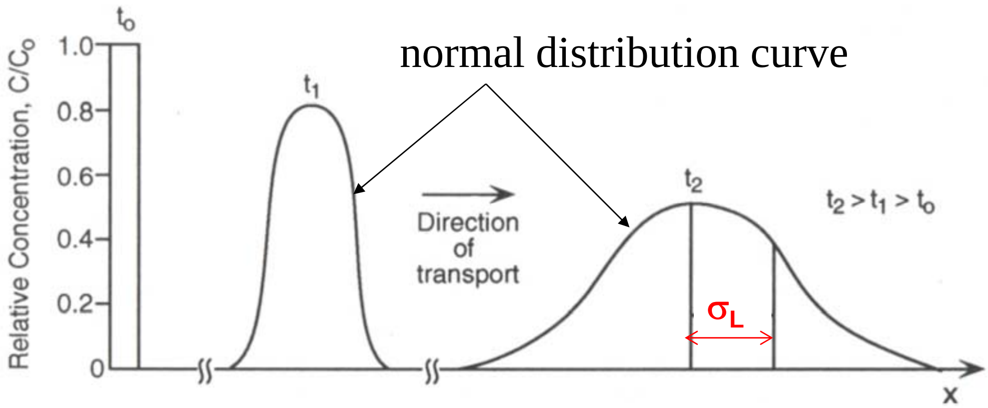
$$\frac{C(x,t)}{C_0} = \frac{1}{2\sqrt{\pi D_m t}} \exp\left(\frac{-x^2}{4D_m t}\right) \quad (25)$$

NDC for μ=0: $$\phi(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{x^2}{2\sigma^2}\right] \quad (26)$$

from (25) and (26) $$2D_m t = \sigma^2, \quad D_m = \sigma^2 / 2t \quad (27)$$

Transport of instantaneous (pulse) source contaminant

(a) 1D



(b) 2D